Serdica
Mathematical Journal
Сердика
Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **35** (2009), 25–60

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DISPERSION PHENOMENA IN DUNKL-SCHRÖDINGER EQUATION AND APPLICATIONS*

H. Mejjaoli

Communicated by I. D. Iliev

Abstract. In this paper, we study the Schrödinger equation associated with the Dunkl operators, we study the dispersive phenomena and we prove the Strichartz estimates for this equation. Some applications are discussed.

1. Introduction. Strichartz estimate is a very interesting topic in the field of dispersive-type partial differential equations. It has wide applications in many other topics, such as well-posedness of initial value problems, regularity of solutions, large-time behavior of solutions, and so on. This topic has a long history starting with seminal paper of Segal [14] and generally goes under the name of Strichartz inequalities after the fundamental paper of Strichartz [15] drawing the connection to the restriction theorems of Tomas and Stein. Standard references on the subject are [6] and [8].

2000 *Mathematics Subject Classification:* 35Q55,42B10.

Key words: Dunkl-Schrödinger equation, Strichartz estimates.

*Thanks to Professor K. Trimèche for his help and encouragement. Thanks is also due to the referee for his suggestions and comments.

In this paper we consider the Dunkl operators T_j, $j = 1, \dots, d$, which are the differential-difference operators introduced by Dunkl in [2] and called Dunkl operators in the literature. These operators are very important in pure Mathematics and in Physics. They provide a useful tool in the study of special functions with root systems (see [3]).

Dunkl in [4] (see also [7]) has studied a Fourier transform $\mathcal{F}_D$ associated with the Dunkl operators, called Dunkl transform defined by

$$\mathcal{F}_D f(x) = \int_{\mathbb{R}^d} K(-ix, y) f(y) \omega_k(y) dy,$$

where K represents is the Dunkl kernel and ω_k is a weight function.

The aim of this paper is to study the Schrödinger equation associated with the Dunkl operators (henceforth called by the Dunkl-Schrödinger equation). Also, the paper establishes the Strichartz-type Schrödinger equation estimates with applications. More precisely, we prove that, for all g in $\mathcal{S}'(\mathbb{R}^d)$, the following problem

$$(S) \left\{ \begin{array}{lcl} \partial_t u - i \triangle_k u & = & 0 \\ u_{|t=0} & = & g, \end{array} \right.$$

(where $\triangle_k = \sum_{j=1}^{d} T_j^2$ is the Dunkl Laplace operator) has a unique solution u in $\mathcal{S}'(\mathbb{R}^{d+1})$ given by

$$u_t(\cdot) = \frac{1}{c_k |t|^{\gamma + \frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4} sgn\, t} e^{i\frac{\|\cdot\|^2}{4t}} \left[\mathcal{F}_D (e^{i\frac{\|\cdot\|^2}{4t}} g) \right] \left(\frac{\cdot}{2t} \right), \quad t \neq 0.$$

We also study the solution of this problem when g belongs to $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{E}'(\mathbb{R}^d)$, or the Dunkl Sobolev spaces $H_k^s(\mathbb{R}^d)$. As consequences, we establish the dispersive estimates for the Dunkl-Schrödinger equation. Moreover, Strichartz-type Schrödinger estimates are proved and both problems of well posedness and the scattering theory associated with the non linear Dunkl-Schrödinger equations are described.

The paper is organized as follows. In Section 2, we recall the main results about the harmonic analysis associated with the Dunkl operators. We introduce in Section 3 the Dunkl-Schrödinger equation. In the same section, we prove that the problem (S) has a unique solution if the initial data g belong to $\mathcal{S}'(\mathbb{R}^d)$, and we present properties of solution when the initial data g belong respectively to the spaces $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{E}'(\mathbb{R}^d)$, and the Dunkl-Sobolev spaces. In Section 4, motivated by the work of Keel and Tao [8] we describe the dispersion phenomena associated for the Dunkl-Schrödinger equation. Finally, Section 5 is devoted to some applications. Namely, we establish the Strichartz estimates for the Dunkl-Schrödinger equation. Besides, we introduce a class of nonlinear Schrödinger equations associated with the Dunkl operators. In this regard, we study local and global well-posedness and scattering theory associated with these equations.

Throughout this paper, C indicates a positive constant not necessarily the same in each occurrence.

2. Preliminaries. This section gives an introduction to the theory of Dunkl operators, Dunkl transform and Dunkl convolution. Main references are [2, 3, 4, 7, 12, 13, 16, 17, 18].

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle\cdot,\cdot\rangle$ and $\|x\|=\sqrt{\langle x,x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflection $\sigma_\alpha, \alpha \in R$, generates a finite group $W \subset O(d)$, called the reflection group associated with R. We fix a positive root system $R_+ = \{\alpha \in R \,/ \langle \alpha, \beta\rangle > 0\}$ for some $\beta \in \mathbb{R}^d\backslash \bigcup\limits_{\alpha\in R} H_\alpha$. We will assume that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in R_+$. A function $k : R \longrightarrow \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha). \tag{2.2}$$

Throughout this paper, we will assume that the multiplicity is non-negative, that is $k(\alpha) \geq 0$ for all $\alpha \in R$. We write $k \geq 0$ for short. Moreover, let ω_k denote the weight function

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} \exp(-\|x\|^2)\omega_k(x)\,dx\right)^{-1}. \tag{2.4}$$

Notations. We denote by
— $C(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$.
— $C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$.
— $C_b^p(\mathbb{R}^d)$ the space of bounded functions of class C^p.
— $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.
— $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
— $D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which have compact support.
— $\mathcal{S}'(\mathbb{R}^d)$ the space of temperate distributions on $\mathbb{R}^d$. It is the topological dual of $\mathcal{S}(\mathbb{R}^d)$.

The Dunkl operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d). \tag{2.5}$$

Some properties of the T_j, $j = 1, \dots, d$, are given in the following list:

For all f and g in $C^1(\mathbb{R}^d)$ with at least one of them W-invariant, we have

$$T_j(fg) = (T_j f)g + f(T_j g), \quad j = 1, \dots, d. \tag{2.6}$$

For f in $C_b^1(\mathbb{R}^d)$ and g in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x)\, dx = -\int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x)\, dx, \quad j = 1, \dots, d. \tag{2.7}$$

We define the Dunkl-Laplace operator on $\mathbb{R}^d$ by

$$\triangle_k f(x) = \sum_{j=1}^d T_j^2 f(x) = \triangle f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \left[\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2} \right]. \tag{2.8}$$

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) &= y_j u(x,y), \quad j = 1, \dots, d, \\ u(0,y) &= 1, \end{cases} \tag{2.9}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. The Dunkl kernel possesses the following properties:

i) For $z, t \in \mathbb{C}^d$, we have $K(z,t) = K(t,z)$; $K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x,z)| \leq \|x\|^{|\nu|} \exp(\|x\| \, \|\mathrm{Re} z\|),$$

with

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \quad \text{and} \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular for all $x, y \in \mathbb{R}^d$:

$$|K(-ix, y)| \leq 1.$$

iii) The function $K(x,z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \tag{2.10}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [12]).

The Dunkl intertwining operator V_k is the operator from $C(\mathbb{R}^d)$ into itself given by

$$V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad \text{for all } x \in \mathbb{R}^d, \tag{2.11}$$

where μ_x is the measure given by the relation (2.10) (see [12]). In particular, we have

$$K(x,z) = V(e^{\langle \cdot, z\rangle})(x), \quad \text{for all } x \in \mathbb{R}^d, \ \text{ and } z \in \mathbb{C}^d.$$

In [3], Dunkl proved that V_k is a linear isomorphism from the space of homogeneous polynomials $\mathcal{P}_n$ on $\mathbb{R}^d$ of degree n into itself satisfying the relations

$$\begin{cases} T_j V_k &= V_k \dfrac{\partial}{\partial x_j}, \quad j = 1, \ldots, d \\ V_k(1) &= 1. \end{cases} \tag{2.12}$$

Trimèche has proved in [17] that the operator V_k can be extended to a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ into itself satisfying the relations (2.12).

Notation. We denote by $L_k^p(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{aligned} \|f\|_{L_k^p(\mathbb{R}^d)} &:= \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x) \, dx \right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad 1 \leq p < +\infty, \\ \|f\|_{L_k^\infty(\mathbb{R}^d)} &:= \operatorname{ess} \sup_{x \in \mathbb{R}^d} |f(x)| < +\infty. \end{aligned}$$

The Dunkl transform of a function f in $L^1_k(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x)K(-iy,x)\omega_k(x)dx, \quad \text{for all } y \in \mathbb{R}^d. \tag{2.13}$$

In the following we give some properties of this transform (see [4] and [7]).

i) For f in $L^1_k(\mathbb{R}^d)$ we have

$$\|\mathcal{F}_D(f)\|_{L^\infty_k(\mathbb{R}^d)} \leq \|f\|_{L^1_k(\mathbb{R}^d)}. \tag{2.14}$$

ii) For f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\mathcal{F}_D(T_j f)(y) = iy_j\mathcal{F}_D(f)(y), \quad \text{for all } j = 1,\dots,d \text{ and } y \in \mathbb{R}^d. \tag{2.15}$$

Proposition 1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathcal{S}(\mathbb{R}^d)$*

$$\overline{\mathcal{F}_D}(f)(y) = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\mathcal{F}_D(f)(-y), \quad y \in \mathbb{R}^d, \tag{2.16}$$

we have

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id.$$

Proposition 2. i) *Plancherel formula for $\mathcal{F}_D$.*
For all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2\omega_k(x)\,dx = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2\omega_k(\xi)\,d\xi. \tag{2.17}$$

ii) *Plancherel theorem for $\mathcal{F}_D$.*
The renormalized Dunkl transform $f \to 2^{-(\gamma+\frac{d}{2})}c_k\mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

Definition 1. *Let y be in $\mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ by*

$$\mathcal{F}_D(\tau_y f)(x) = K(ix,y)\mathcal{F}_D(f)(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{2.18}$$

Proposition 3. i) *The operator τ_y, $y \in \mathbb{R}^d$, can also be defined on $\mathcal{E}(\mathbb{R}^d)$ by*

$$\tau_y f(x) = (V_k)_x(V_k)_y[(V_k)^{-1}(f)(x+y)], \quad \text{for all } x \in \mathbb{R}^d. \tag{2.19}$$

(*See* [18]).

ii) *If* $f(x) = F(\|x\|)$ *in* $\mathcal{E}(\mathbb{R}^d)$, *then we have*

$$\tau_y f(x) = V_k \left[F(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, \cdots \rangle}) \right](x), \quad \text{for all } x \in \mathbb{R}^d.$$

(*See* [13]).

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [16] and [18]).

Definition 2. *The Dunkl convolution product of* f *and* g *in* $\mathcal{S}(\mathbb{R}^d)$ *is the function* $f *_D g$ *defined by*

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.20}$$

This convolution is commutative and associative and satisfies the following properties. (See [16]).

Proposition 4. i) *For* f *and* g *in* $D(\mathbb{R}^d)$ (*resp.* $\mathcal{S}(\mathbb{R}^d)$) *the function* $f *_D g$ *belongs to* $D(\mathbb{R}^d)$ (*resp.* $\mathcal{S}(\mathbb{R}^d)$) *and we have*

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y), \quad \text{for all } y \in \mathbb{R}^d. \tag{2.21}$$

ii) *Let* $1 \leq p, q, r \leq \infty$, *such that* $\dfrac{1}{p} + \dfrac{1}{q} - \dfrac{1}{r} = 1$. *If* f *is in* $L_k^p(\mathbb{R}^d)$ *and* g *is a radial element of* $L_k^q(\mathbb{R}^d)$, *then* $f *_D g \in L_k^r(\mathbb{R}^d)$ *and we have*

$$\|f *_D g\|_{L_k^r(\mathbb{R}^d)} \leq \|f\|_{L_k^p(\mathbb{R}^d)} \|g\|_{L_k^q(\mathbb{R}^d)}. \tag{2.22}$$

Definition 3. *The Dunkl transform of a distribution* τ *in* $\mathcal{S}'(\mathbb{R}^d)$ *is defined by*

$$\langle \mathcal{F}_D(\tau), \phi \rangle = \langle \tau, \mathcal{F}_D(\phi) \rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^d). \tag{2.23}$$

Proposition 5. *The Dunkl transform* $\mathcal{F}_D$ *is a topological isomorphism from* $\mathcal{S}'(\mathbb{R}^d)$ *onto itself.*

For f in $L_k^p(\mathbb{R}^d)$, we define the tempered distribution T_f associated with f by

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}^d} f(x)\phi(x)\omega_k(x)dx, \ \phi \in \mathcal{S}(\mathbb{R}^d). \tag{2.24}$$

Definition 4. i) *The Dunkl transform of a distribution τ in $\mathcal{S}'(\mathbb{R}^d)$ is defined by*

$$\langle \mathcal{F}_D(\tau), \phi\rangle = \langle \tau, \mathcal{F}_D(\phi)\rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d). \tag{2.25}$$

ii) *We define $\overline{\mathcal{F}_D}$ on $\mathcal{S}'(\mathbb{R}^d)$ by similar formula.*

iii) *The Dunkl transform of f in $L^p_k(\mathbb{R}^d)$ denoted also by $\mathcal{F}_D(f)$, is defined by*

$$\langle \mathcal{F}_D(f), \phi\rangle = \langle \mathcal{F}_D(T_f), \phi\rangle = \langle T_f, \mathcal{F}_D(\phi)\rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Thus from (2.24) we have

$$\langle \mathcal{F}_D(f), \phi\rangle = \int_{\mathbb{R}^d} f(x)\mathcal{F}_D(\phi)(x)\omega_k(x)dx. \tag{2.26}$$

Definition 5. *The Dunkl convolution product of a distribution S in $\mathcal{S}'(\mathbb{R}^d)$ and a function ϕ in $\mathcal{S}(\mathbb{R}^d)$ is the function $S *_D \phi$ defined by*

$$\forall\, x \in \mathbb{R}^d, \quad S *_D \phi(x) = \langle S_y, \tau_{-y}\phi(x)\rangle. \tag{2.27}$$

Proposition 6. i) *Let f be in $L^p_k(\mathbb{R}^d)$, $p \in [1, +\infty]$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then the distribution $T_f *_D \phi$ is given by the function $f *_D \phi$. If we assume that f is even for $d = 1$ and radial for $d \geq 2$, then $T_f *_D \phi \in L^p_k(\mathbb{R}^d)$.*

ii) *Assume that $f \in L^p_k(\mathbb{R}^d)$, $p \in [1, +\infty]$ is even for $d = 1$ and radial for $d \geq 2$ and ϕ_1,ϕ_2 in $\mathcal{S}(\mathbb{R}^d)$. Then we have*

$$\langle T_f *_D \phi_1, \phi_2\rangle = \langle T_{\check{f}}, \phi_1 *_D \check{\phi}_2\rangle, \tag{2.28}$$

where $\check{h}(x) = h(-x)$.

iii) *Let $f \in L^p_k(\mathbb{R}^d)$, $p \in [1, +\infty]$ be even for $d = 1$ and radial for $d \geq 2$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then we have*

$$\mathcal{F}_D(T_f *_D \phi) = \mathcal{F}_D(T_f)\, \mathcal{F}_D(\phi). \tag{2.29}$$

Proof. i) Let f be in $L^p_k(\mathbb{R}^d)$, $p \in [1, +\infty]$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. We have from (2.24)

$$\begin{aligned} \forall\, x \in \mathbb{R}^d,\ T_f *_D \phi(x) &= \langle (T_f)_y, \tau_x\phi(-y)\rangle, \\ &= f *_D \phi(x). \end{aligned}$$

Now we assume that f is even for $d=1$ and radial for $d\geq 2$. Applying Proposition 4 ii), we obtain that $T_f *_D \phi \in L^p_k(\mathbb{R}^d)$.

ii) Let $f \in L^p_k(\mathbb{R}^d)$, $p\in[1,+\infty]$ be even for $d=1$ and radial for $d\geq 2$ and ϕ_1,ϕ_2 in $\mathcal{S}(\mathbb{R}^d)$. Then from Fubini-Tonelli's theorem the function $(x,y)\mapsto f(-y)\tau_x\phi_1(y)\phi_2(x)$ is integrable on $\mathbb{R}^d\times\mathbb{R}^d$ with respect to the measure $\omega_k(y)dy\,\omega_k(x)dx$. Thus from Fubini's theorem we obtain

$$\begin{aligned}\langle T_f *_D \phi_1,\phi_2\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(-y)\tau_x\phi_1(y)\phi_2(x)\omega_k(y)dy\omega_k(x)dx\\ &= \int_{\mathbb{R}^d} f(-y)\left(\int_{\mathbb{R}^d}\tau_y\phi_1(x)\phi_2(x)\omega_k(x)dx\right)\omega_k(y)dy\\ &= \int_{\mathbb{R}^d} f(-y)\phi_1 *_D \check{\phi}_2(y)\omega_k(y)dy\\ &= \langle T_{\check{f}},\phi_1 *_D \check{\phi}_2\rangle.\end{aligned}$$

iii) Let $f\in L^p_k(\mathbb{R}^d)$, $p\in[1,+\infty]$ be even for $d=1$ and radial for $d\geq 2$ and ϕ in $\mathcal{S}(\mathbb{R}^d)$. Then from i) and the relations (2.28)(2.21) we have for any φ belongs to $\mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}\langle \mathcal{F}_D(T_f *_D \phi),\varphi\rangle &= \langle T_f *_D \phi,\mathcal{F}_D(\varphi)\rangle = \langle T_{\check{f}},\phi *_D \check{\mathcal{F}_D(\varphi)}\rangle\\ &= \langle T_f,\mathcal{F}_D(\mathcal{F}_D(\phi)\varphi)\rangle = \langle \mathcal{F}_D(T_f)\mathcal{F}_D(\phi),\varphi\rangle.\end{aligned}$$

Thus we have the result. $\square$

Let τ be in $\mathcal{S}'(\mathbb{R}^d)$. We define the distributions $T_j\tau$, $j=1,\dots,d$, by

$$\langle T_j\tau,\psi\rangle = -\langle\tau,T_j\psi\rangle,\ \text{ for all } \psi\in\mathcal{S}(\mathbb{R}^d). \tag{2.30}$$

Thus we deduce

$$\langle \triangle_k\tau,\psi\rangle = \langle\tau,\triangle_k\psi\rangle,\ \text{ for all } \psi\in\mathcal{S}(\mathbb{R}^d). \tag{2.31}$$

These distributions satisfy the following properties

$$\mathcal{F}_D(T_j\tau) = iy_j\mathcal{F}_D(\tau),\quad j=1,\dots,d. \tag{2.32}$$

$$\mathcal{F}_D(\triangle_k\tau) = -\|y\|^2\mathcal{F}_D(\tau). \tag{2.33}$$

Below, if $f\in L^p_k(\mathbb{R}^d)$, the distribution T_f given by the relation (2.24), is noted by f.

3. Dunkl-Schrödinger equation.

Notations. We denote by:

— $D'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$. It is the topological dual of $D(\mathbb{R}^d)$.

— $H^s_k(\mathbb{R}^d)$ the Dunkl-Sobolev spaces defined for $s \in \mathbb{R}$, by

$$\Big\{u \in \mathcal{S}'(\mathbb{R}^d) : (1+\|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_D(u) \in L^2_k(\mathbb{R}^d)\Big\}.$$

We provide this space with the scalar product

$$\langle u, v\rangle_{s,k} = \int_{\mathbb{R}^d}(1+\|\xi\|^2)^s\mathcal{F}_D(u)(\xi)\overline{\mathcal{F}_D(v)(\xi)}\omega_k(\xi)d\xi$$

and the norm

$$\|u\|^2_{s,k} = \langle u, u\rangle_{s,k}.$$

— $H^s_{loc,k}(\mathbb{R}^d) = \Big\{u \in D'(\mathbb{R}^d) : \phi u \in H^s_k(\mathbb{R}^d),\ \forall\, \phi \in D(\mathbb{R}^d)\Big\}$.

— $C(\mathbb{R}; H^s_k(\mathbb{R}^d))$ the space of continuous functions from $\mathbb{R}$ into $H^s_k(\mathbb{R}^d)$.

— $\mathcal{E}(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ the space of C^∞ functions from $\mathbb{R}$ into $\mathcal{S}(\mathbb{R}^d)$.

— $\mathcal{E}(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ the space of C^∞ functions from $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^d)$.

We consider the following equation where the unknown is a function u (with complex values) of $(t, x) \in \mathbb{R}\times\mathbb{R}^d$

$$(S)\left\{\begin{array}{lcl} \partial_t u - i\triangle_k u &=& 0\\ u_{|t=0} &=& g. \end{array}\right.$$

Theorem 1. *Let g be in $\mathcal{S}'(\mathbb{R}^d)$. There exists a unique solution u in $\mathcal{E}(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ such that*

$$(S)\left\{\begin{array}{lcl} \partial_t u - i\triangle_k u &=& 0, \quad \text{in} \quad D'(\mathbb{R}\times\mathbb{R}^d)\\ u_{|t=0} &=& g. \end{array}\right.$$

Proof. First we prove the existence. For $t \in \mathbb{R}$, we put

$$u_t = \overline{\mathcal{F}_D}(e^{-it\|\xi\|^2}\mathcal{F}_D(g)). \tag{3.34}$$

From (2.25) we have

$$\langle u_t, \varphi\rangle = \langle \mathcal{F}_D(g), e^{-it\|\xi\|^2}\overline{\mathcal{F}_D}(\varphi)\rangle.$$

Thus we deduce that $u_t \in \mathcal{E}(\mathbb{R};\mathcal{S}'(\mathbb{R}^d))$, and $\mathcal{F}_D(u_t) \in \mathcal{E}(\mathbb{R};\mathcal{S}'(\mathbb{R}^d))$.
We recall that u is defined by

$$\langle u,\psi\rangle = \int_{\mathbb{R}} \langle u_t, \psi(t,\cdot)\rangle dt,\ \ \psi \in \mathcal{S}(\mathbb{R}^{d+1}).$$

Then for any ψ in $\mathcal{S}(\mathbb{R}^{d+1})$, we have from (2.31)

$$\begin{aligned}
\langle \partial_t u - i\triangle_k u, \psi\rangle &= -\langle u, \partial_t\psi + i\triangle_k\psi\rangle = -\int_{\mathbb{R}} \langle u_t, \partial_t\psi(t,\cdot) + i\triangle_k\psi(t,\cdot)\rangle dt \\
&= -\int_{\mathbb{R}} \left\langle \mathcal{F}_D(u_t), \overline{\mathcal{F}_D}\left(\partial_t\psi(t,\cdot) + i\triangle_k\psi(t,\cdot)\right)\right\rangle dt \\
&= -\int_{\mathbb{R}} \left\langle e^{-it\|\cdot\|^2}\mathcal{F}_D(g), (\partial_t - i\|\cdot\|^2)\overline{\mathcal{F}_D}\psi(t,\cdot)\right\rangle dt.
\end{aligned}$$

But

$$\partial_t\Big(e^{-it\|\xi\|^2}\overline{\mathcal{F}_D}\psi(t,\xi)\Big) = \Big[(\partial_t - i\|\xi\|^2)\overline{\mathcal{F}_D}\psi(t,\xi)\Big]e^{-it\|\xi\|^2}.$$

Thus

$$\begin{aligned}
\langle \partial_t u_t - i\triangle_k u, \psi\rangle &= -\int_{\mathbb{R}} \left\langle \mathcal{F}_D(g), \partial_t\left(e^{-it\|\cdot\|^2}\overline{\mathcal{F}_D}\psi(t,\cdot)\right)\right\rangle dt \\
&= -\int_{\mathbb{R}} \partial_t\left\langle \mathcal{F}_D(g), e^{-it\|\cdot\|^2}\overline{\mathcal{F}_D}\psi(t,\cdot)\right\rangle dt = 0.
\end{aligned}$$

Thus we have proved that u is solution of (S).

Now we prove the uniqueness, or equivalently that $u \equiv 0$ is the unique solution of problem

$$\begin{cases} \partial_t u - i\triangle_k u &= 0, \quad \text{in} \quad \mathcal{E}(\mathbb{R};\mathcal{S}'(\mathbb{R}^d)) \\ u_{|t=0} &= 0. \end{cases}$$

Indeed for all ψ in $\mathcal{S}(\mathbb{R}^{d+1})$ we have

$$0 = \langle \partial_t u - i\triangle_k u, \psi\rangle = -\int_{\mathbb{R}} \langle u_t, (\partial_t + i\triangle_k)\psi(t,\cdot)\rangle dt.$$

But

$$\frac{d}{dt}\langle u_t, \psi(t,\cdot)\rangle = \langle u_t^{(1)}, \psi(t,\cdot)\rangle + \langle u_t, \partial_t\psi(t,\cdot)\rangle,$$

hence

$$0 = -\int_{\mathbb{R}} \frac{d}{dt}\langle u_t, \psi(t,\cdot)\rangle dt + \int_{\mathbb{R}} \Big[\langle u_t^{(1)}, \psi(t,\cdot)\rangle - i\langle u_t, \triangle_k\psi(t,\cdot)\rangle\Big] dt. \tag{3.35}$$

As $\psi(-\infty,\cdot)=\psi(+\infty,\cdot)=0$, we then obtain

$$\int_{\mathbb{R}}\Big[\langle u_t^{(1)},\psi(t,\cdot)\rangle - i\langle u_t,\triangle_k\psi(t,\cdot)\rangle\Big]dt=0. \tag{3.36}$$

Moreover, using that $\mathcal{F}_D(u_t^{(1)})=(\mathcal{F}_D(u_t))^{(1)}$ and the relations (3.36), (2.15) we deduce

$$\int_{\mathbb{R}}\Big[\langle(\mathcal{F}_D(u_t))^{(1)},\overline{\mathcal{F}_D}\psi(t,\cdot)\rangle + i\langle\mathcal{F}_D(u_t),\|\cdot\|^2\overline{\mathcal{F}_D}\psi(t,\cdot)\rangle\Big]\,dt=0, \quad \forall\,\psi\in\mathcal{S}(\mathbb{R}^{d+1}). \tag{3.37}$$

If we take ψ such that $\overline{\mathcal{F}_D}\psi(t,\xi)=e^{it\|\xi\|^2}\varphi(\xi)\chi(t)$ with φ in $\mathcal{S}(\mathbb{R}^d)$, χ in $\mathcal{S}(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}}\Big[\langle(\mathcal{F}_D(u_t))^{(1)},e^{it\|\cdot\|^2}\varphi\rangle + i\langle\mathcal{F}_D(u_t),\|\cdot\|^2e^{it\|\cdot\|^2}\varphi\rangle\Big]\,\chi(t)dt=0, \quad \forall\,\chi\in\mathcal{S}(\mathbb{R}). \tag{3.38}$$

Thus we deduce that

$$\begin{aligned}\frac{d}{dt}\langle\mathcal{F}_D(u_t),e^{it\|\cdot\|^2}\varphi\rangle &= \langle(\mathcal{F}_D(u_t))^{(1)},e^{it\|\cdot\|^2}\varphi\rangle + i\langle\mathcal{F}_D(u_t),\|\cdot\|^2e^{it\|\cdot\|^2}\varphi\rangle\\ &=0, \quad \forall\,\varphi\in\mathcal{S}(\mathbb{R}^d).\end{aligned} \tag{3.39}$$

Hence for all φ in $\mathcal{S}(\mathbb{R}^d)$, the function $t\mapsto\langle\mathcal{F}_D(u_t),e^{it\|\cdot\|^2}\varphi\rangle$ is constant.
Finally, since $u_0=0$ then

$$\langle\mathcal{F}_D(u_t),e^{it\|\cdot\|^2}\varphi\rangle=\langle\mathcal{F}_D(u_0),\varphi\rangle=0,\ \forall\,t\in\mathbb{R};\ \forall\,\varphi\in\mathcal{S}(\mathbb{R}^d).$$

From this we deduce that $u=0$. $\square$

Proposition 7. *Let g be in $\mathcal{S}(\mathbb{R}^d)$. The solution u given by Theorem 1 belongs to $\mathcal{E}(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$ and it is given by :*

$$u(t,x)=\frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d}K(ix,\xi)e^{-it\|\xi\|^2}\mathcal{F}_D(g)(\xi)\omega_k(\xi)d\xi. \tag{3.40}$$

Proof. Let g be in $\mathcal{S}(\mathbb{R}^d)$. Formula (3.34) gives that u_t belongs to $\mathcal{S}(\mathbb{R}^d)$ and that $u_t(x)$ is equal to the second member term of (3.40). Thus it is easy to see that the function $(t,x)\mapsto u_t(x)$ is C^∞ on $\mathbb{R}\times\mathbb{R}^d$. Below we will write $u(t,x)$ instead of $u_t(x)$.

Moreover from Proposition 1 and the relations (2.9), (2.6), (2.7) we have,

$$
\begin{aligned}
(ix)^\nu T^\mu u(t,x) &= \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d} T_\xi^\nu K(ix,\xi)e^{-it\|\xi\|^2}(i\xi)^\mu \mathcal{F}_D(g)(\xi)\omega_k(\xi)d\xi \\
&= \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d} K(ix,\xi)(-T_\xi)^\nu \left[e^{-it\|\xi\|^2}(i\xi)^\mu \mathcal{F}_D(g)(\xi)\right]\omega_k(\xi)d\xi \\
&= \int_{\mathbb{R}^d} K(ix,\xi)h_{\nu,\mu}(t,\xi)e^{-it\|\xi\|^2}\omega_k(\xi)d\xi, \qquad (3.41)
\end{aligned}
$$

where $h_{\nu,\mu}$ is an element of $\mathcal{E}(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$.

Using (3.41) and dominated convergence theorem it is easy to see that the solution u belongs to $\mathcal{E}(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$. $\square$

Proposition 8. *Let g be in $H_k^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, and let the solution u given by Theorem 1 belong to $C(\mathbb{R};H_k^s(\mathbb{R}^d))$. For p in $\mathbb{N}$, $(u_t^{(p)})\in C(\mathbb{R};H_k^{s-2p}(\mathbb{R}^d))$ and we have*

$$
\left\{\begin{array}{lcl}
\|u_t\|_{s,k} &=& \|g\|_{s,k},\ \forall\, t\in\mathbb{R}\\
\|u_t^{(p)}\|_{s-2p,k} &\leq& C_p\|g\|_{s,k},\ \forall\, t\in\mathbb{R};\ \forall\, p\in\mathbb{N}^*.
\end{array}\right. \qquad (3.42)
$$

P r o o f. Formula (3.34) gives that, for all t in $\mathbb{R}$,

$$\mathcal{F}_D(u_t) = e^{-it\|\xi\|^2}\mathcal{F}_D(g).$$

Thus it is easy to deduce (3.42).

Now we will prove that for p in $\mathbb{N}$, $(u_t^{(p)})$ belongs to $C(\mathbb{R};H_k^{s-2p}(\mathbb{R}^d))$. Indeed, let $(t_n)_n$ be a sequence that converge to t_0 in $\mathbb{R}$. We have

$$\|u_{t_n}-u_{t_0}\|_{s,k}^2 = \int_{\mathbb{R}^d}(1+\|\xi\|^2)^s|e^{-it_n\|\xi\|^2}-e^{-it_0\|\xi\|^2}|^2|\mathcal{F}_D(g)(\xi)|^2\omega_k(\xi)d\xi.$$

The dominate convergence theorem gives that $\lim_{n\to\infty}\|u_{t_n}-u_{t_0}\|_{s,k}^2=0$. On the other hand, from (3.34) we have

$$\mathcal{F}_D(u_t^{(p)}) = (-i\|\xi\|^2)^p e^{-it\|\xi\|^2}\mathcal{F}_D(g).$$

From this we obtain

$$\|u_{t_n}^{(p)}-u_{t_0}^{(p)}\|_{s,k}^2 = \int_{\mathbb{R}^d}(1+\|\xi\|^2)^s|e^{-it_n\|\xi\|^2}-e^{-it_0\|\xi\|^2}|^2\|\xi\|^p|\mathcal{F}_D(g)(\xi)|^2\omega_k(\xi)d\xi.$$

Thus the dominated convergence theorem gives the result. $\square$

Theorem 2. *Let g be in $\mathcal{S}'(\mathbb{R}^d)$. The solution u given by Theorem 1 can be written, for $t \neq 0$, as*

$$u_t(\cdot) = \frac{1}{c_k |t|^{\gamma+\frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4} sgn\, t} e^{i\frac{\|\cdot\|^2}{4t}} \left[\mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}} g)\right]\left(\frac{\cdot}{2t}\right). \tag{3.43}$$

P r o o f. First we prove this formula for $g \in D(\mathbb{R}^d)$. Indeed, the relation (3.34) can be written as

$$u(t,x) = \left[\overline{\mathcal{F}_D}(e^{-it\|\xi\|^2}) *_D g\right](x). \tag{3.44}$$

Now we are going to calculate

$$\overline{\mathcal{F}_D}(e^{-it\|\xi\|^2}).$$

From [13, p. 2424], we have for any positive a

$$\overline{\mathcal{F}_D}(e^{-a\|\xi\|^2}) = \frac{1}{c_k a^{\gamma+\frac{d}{2}}} e^{-\frac{\|\cdot\|^2}{4a}}.$$

Let us observe that for $z \in \mathbb{C}$, $z = |z|e^{i\theta}$ with positive real part, by taking the branch $z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, the two functions

$$z \mapsto \overline{\mathcal{F}_D}(e^{-z\|\xi\|^2}) \quad \text{and} \quad z \mapsto \frac{1}{c_k z^{\gamma+\frac{d}{2}}} e^{-\frac{\|\cdot\|^2}{4z}}$$

are holomorphic on the domain $Re z > 0$. As they coincide on the real axis, they coincide in the whole domain. Now, if t positive, considering a sequence of z_n with positive real part which tends to it, we get, as the Dunkl transform is continuous on tempered distributions that

$$\overline{\mathcal{F}_D}(e^{-it\|\xi\|^2}) = \frac{1}{c_k t^{\gamma+\frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4}} e^{i\frac{\|\cdot\|^2}{4t}}.$$

In the case when t is negative, we have, following the same way, that

$$\overline{\mathcal{F}_D}(e^{-it\|\xi\|^2}) = \frac{1}{c_k |t|^{\gamma+\frac{d}{2}}} e^{i(d+2\gamma)\frac{\pi}{4}} e^{i\frac{\|\cdot\|^2}{4t}}.$$

Thus, we have

$$\overline{\mathcal{F}_D}(e^{-it\|\xi\|^2}) = \frac{1}{c_k |t|^{\gamma+\frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4} sgn\, t} e^{i\frac{\|\cdot\|^2}{4t}}.$$

Hence, from (3.44) we obtain

$$u(t,x) = \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4} sgn\, t}(e^{i\frac{\|\cdot\|^2}{4t}} *_D g)(x). \tag{3.45}$$

On the other hand, since $g \in D(\mathbb{R}^d)$, we have

$$\begin{aligned}\left(e^{i\frac{\|\cdot\|^2}{4t}} *_D g\right)(x) &= \int_{\mathbb{R}^d} K\left(\frac{-ix}{2t}, y\right) e^{i\frac{\|x\|^2+\|y\|^2}{4t}} g(y)\omega_k(y)dy \\ &= e^{i\frac{\|x\|^2}{4t}} \mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}} g)\left(\frac{x}{2t}\right).\end{aligned}$$

Hence

$$u(t,x) = \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4} sgn\, t} e^{i\frac{\|x\|^2}{4t}} \left[\mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}} g)\right]\left(\frac{x}{2t}\right). \tag{3.46}$$

Thus formula (3.43) is proved in the case $g \in D(\mathbb{R}^d)$.

We assume now g is in $\mathcal{S}'(\mathbb{R}^d)$. There exists $(g_p)_p$ in $D(\mathbb{R}^d)$ such that g_p converge to g in $\mathcal{S}'(\mathbb{R}^d)$. Let u_p be the solution of problem (S) with initial data g_p. From Proposition 7, we have $u_p \in \mathcal{E}(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ and according to the first case, u_p is given by formula (3.46). Consider the right member of (3.46).

One has

$$e^{i\frac{\|\cdot\|^2}{4t}} g_p \to e^{i\frac{\|\cdot\|^2}{4t}} g \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d),$$

therefore

$$\mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}} g_p) \circ A_t \to \mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}} g) \circ A_t \text{ in } \mathcal{S}'(\mathbb{R}^d),$$

where $A_t : \mathbb{R}^d \to \mathbb{R}^d$ is the application $x \mapsto \dfrac{x}{2t}$. As the multiplication by $e^{i\frac{\|x\|^2}{4t}}$ is continuous $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, we deduce that the right-hand side of (3.46) taken with g_p converges to the same expression with g, in $\mathcal{S}'(\mathbb{R}^d)$. On the other hand, according to the uniqueness in Theorem 1, the solution u_p of the problem (S), with initial data g_p is given by $(u_p)_t = \overline{\mathcal{F}_D}(e^{-it\|\cdot\|^2}\mathcal{F}_D(g_p))$, as $\mathcal{F}_D(g_p)$ converges to $\mathcal{F}_D(g)$ in $\mathcal{S}'(\mathbb{R}^d)$ and $e^{-it\|\cdot\|^2}\mathcal{F}_D(g_p)$ converges to $e^{-it\|\cdot\|^2}\mathcal{F}_D(g)$ in $\mathcal{S}'(\mathbb{R}^d)$, $(u_p)_t$ converges to u_t in $\mathcal{S}'(\mathbb{R}^d)$ where u is the solution of (S) with initial data g. Therefore the formula (3.46) implies (3.43). □

Corollary 1. *If g is in $L^1_k(\mathbb{R}^d)$, then we have the dispersion estimate:*

$$\|u_t\|_{L^\infty_k(\mathbb{R}^d)} \le \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}}\|g\|_{L^1_k(\mathbb{R}^d)}. \tag{3.47}$$

P r o o f. The previous Theorem and relation (2.14) give the result. $\square$

Notation. We denote by $\mathcal{E}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$ with compact support.

Another important application of Theorem 1 is the following.

Corollary 2. i) *If $g \in \mathcal{E}'(\mathbb{R}^d)$ and u is the solution of problem (S), then, $u_t \in \mathcal{E}(\mathbb{R}^d)$, for all $t \neq 0$.*

ii) *Let $g(x) = e^{-i\lambda\|x\|^2}$, with $\lambda > 0$. Then*

$$u_{\frac{1}{4\lambda}} = \frac{1}{c_k}\lambda^{\gamma+\frac{d}{2}} e^{-i(\gamma+\frac{d}{2})\frac{\pi}{4}}\delta.$$

P r o o f. i) If $g \in \mathcal{E}'(\mathbb{R}^d)$, then $e^{i\frac{\|\cdot\|^2}{4t}} g \in \mathcal{E}'(\mathbb{R}^d)$, and from Paley-Wiener Theorem for the distribution with compact support (see [18]) we have $\mathcal{F}_D\left(e^{i\frac{\|\cdot\|^2}{4t}} g\right) \in \mathcal{E}(\mathbb{R}^d)$ and formula (3.43) gives i).

ii) For $t = \dfrac{1}{4\lambda}$, we have

$$\mathcal{F}_D\left(e^{i\frac{\|\cdot\|^2}{4t}} g\right) = \mathcal{F}_D(1) = \delta.$$

Then

$$e^{i\frac{\|\cdot\|^2}{4t}} \mathcal{F}_D\left(e^{i\frac{\|\cdot\|^2}{4t}} g\right) = \delta.$$

This gives the result. $\square$

Remark. From the previous Corollary we remark that the regularity of solution, for $t \neq 0$ depends on the behavior at infinity of the initial data and not of its regularity. This phenomenon is known under the name of distribution to infinite speed.

Proposition 9. *Let g be in $L^2_k(\mathbb{R}^d)$ and we assume that, for all $\mu \in \mathbb{N}^d$, we have $x^\mu g \in L^2_k(\mathbb{R}^d)$. Then for $t \neq 0$, $u_t \in \mathcal{E}(\mathbb{R}^d)$, where u is the solution of problem (S).*

P r o o f. By a simple calculation we prove that

$$[\partial_t - i\triangle_k, x_j + 2itT_j] = 0,\ j = 1, \ldots, d,$$

where $[A, B] = AB - BA$.

By iteration of the previous identity we obtain

$$[\partial_t - i\triangle_k, (x + 2itT)^\mu] = 0, \ \forall\, \mu \in \mathbb{N}^d. \tag{3.48}$$

Moreover, using Proposition 8 we obtain that the solution u of system (S) belongs to $C(\mathbb{R}; L^2_k(\mathbb{R}^d))$. Next, we consider $V_\mu(t,x) = (x + 2itT)^\mu u(t,x)$. We have V_μ is an element of $C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ and $V_\mu(0,\cdot)$ belongs to $L^2_k(\mathbb{R}^d)$. On the other hand using the identity (3.48) we obtain $(\partial_t - i\triangle_k)V_\mu = 0$. Hence from Proposition 8 we deduce that,

$$V_\mu \in C(\mathbb{R}; L^2_k(\mathbb{R}^d)), \ \mu \in \mathbb{N}^d. \tag{3.49}$$

As

$$(x + 2itT)^\mu = \sum_{|\beta| \le |\mu|-1} P_{\mu,\beta}(t,x) T^\beta + (2it)^{|\mu|} T^\mu, \tag{3.50}$$

where $P_{\mu,\beta}$ is a polynomial in (t,x). Then using the relations (3.49),(3.50), the fact that u is in $C(\mathbb{R}; L^2_k(\mathbb{R}^d))$ and by induction on μ we obtain that $T^\mu_x u_t$ belongs to $L^2_{loc,k}(\mathbb{R}^d)$, for $t \neq 0$.

Thus, by Theorem 3.3 of [7] we obtain

$$u_t \in H^s_{loc,k}(\mathbb{R}^d), \ s \in \mathbb{N}, \ t \neq 0.$$

As $\bigcap_{s\ge 0} H^s_{loc,k}(\mathbb{R}^d) \subset \mathcal{E}(\mathbb{R}^d)$ (Theorem 3.4 of [7]), the result is proved. □

4. Dispersion phenomena.

Notations. For any interval I of $\mathbb{R}$ (bounded or unbounded) and a Banach space X, we define the mixed space-time $L^q(I;X)$ Banach space of (classes of) measurable functions $u : I \to X$ such that $\|u\|_{L^q(I;X)} < \infty$, with

$$\begin{aligned} \|u\|_{L^q(I;X)} &= \Big(\int_I \|u(t,\cdot)\|^q_X dt\Big)^{\frac{1}{q}}, \quad \text{if } 1 \le q < \infty, \\ \|u\|_{L^\infty(I;X)} &= ess \sup_{t\in I} \|u(t,\cdot)\|_X. \end{aligned}$$

$C(\overline{I};X)$ the space of continuous functions $\overline{I} \to X$. When I is bounded, $C(\overline{I};X)$ is a Banach space with the norm of $L^\infty(I,X)$.

$C_c(I, \mathcal{S}(\mathbb{R}^d))$ is the space of continuous functions from I into $\mathcal{S}(\mathbb{R}^d)$ compactly supported in I, equipped with the topology of uniform convergence on the compact subintervals of I.

Definition 6. *We say that the exponent pair* (q,r) *is* $\frac{d+2\gamma}{2}$*-admissible if* $q,r\geq 2$, $\left(q,r,\frac{d+2\gamma}{2}\right)\neq(2,\infty,1)$ *and*

$$\frac{1}{q}+\frac{d+2\gamma}{2r}\leq\frac{d+2\gamma}{4}. \tag{4.51}$$

If equality holds in (4.51)*, we say that* (q,r) *is sharp* $\frac{d+2\gamma}{2}$*-admissible, otherwise we say that* (q,r) *is nonsharp* $\frac{d+2\gamma}{2}$*-admissible. Note in particular that when* $d+2\gamma>2$ *the endpoint*

$$P=\left(2,\frac{2d+4\gamma}{d+2\gamma-2}\right)$$

is sharp $\frac{d+2\gamma}{2}$*-admissible.*

Theorem 3. *Let* $(U(t))_{t\in\mathbb{R}}$ *be a bounded family of continuous operators on* $L^2_k(\mathbb{R}^d)$ *such that, we have*

$$\|U(t)U^*(t')f\|_{L^\infty_k(\mathbb{R}^d)}\leq\frac{C}{|t-t'|^{\frac{d}{2}+\gamma}}\|f\|_{L^1_k(\mathbb{R}^d)}. \tag{4.52}$$

Then, the estimates

$$\|U(t)u_0\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))}\ \leq\ C\|u_0\|_{L^2_k(\mathbb{R}^d)} \tag{4.53}$$

$$\left\|\int_{\mathbb{R}}U^*(t)f(t,\cdot)dt\right\|_{L^2_k(\mathbb{R}^d)}\ \leq\ C\|f\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))} \tag{4.54}$$

hold for any sharp $\frac{d+2\gamma}{2}$*-admissible exponent* (q,r)*, where* q',r' *are the conjugate exponents of* q *and* r *and* U^* *is the adjoint operator of* U.

Moreover, for any sharp $\frac{d+2\gamma}{2}$*-admissible exponent pairs* (q,r) *and* (q_1,r_1) *we have*

$$\left\|\int_{\mathbb{R}}U(t)U^*(t')f(t',\cdot)dt'\right\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))}\leq C\|f\|_{L^{q_1'}(\mathbb{R};L^{r_1'}_k(\mathbb{R}^d))}. \tag{4.55}$$

Furthermore, if

$$\|U(s)U^*(t)f\|_{L^\infty_k(\mathbb{R}^d)}\leq\frac{C}{(1+|t-s|)^{\frac{d+2\gamma}{2}}}\|f\|_{L^1_k(\mathbb{R}^d)}, \tag{4.56}$$

then (4.53), (4.54) *and* (4.55) *hold for all* $\frac{d+2\gamma}{2}$*-admissible* (q,r) *and* (q_1,r_1).

Before to demonstrate this theorem, we need the next result:

Proposition 10 (Hardy-Littlewood-Sobolev inequality)**.** *Let* β *be in* $]0,d[$*. Then, if* p *and* q *are in* $(1,+\infty)$ *such that*

$$\frac{1}{p}+\frac{\beta}{d}=\frac{1}{q}+1,$$

then, a constant C *exists such that*

$$\|\,\|\cdot\|^{-\beta}*f\|_{L^q(\mathbb{R}^d)}\leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

Proof of Theorem 3. We divide the proof of this theorem in two steps:

1^{st} step: $(q,r)\neq P$. We have

$$\begin{aligned}\|U(t)u_0\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))} &= \sup_{\varphi\in B^{q,r}_k}\int_{\mathbb{R}^{d+1}}U(t)u_0(x)\overline{\varphi(t,x)}dt\omega_k(x)dx\\ &= \sup_{\varphi\in B^{q,r}_k}\langle u_0,\int_{\mathbb{R}}U^*(t)\varphi(t,\cdot)\rangle_{L^2_k(\mathbb{R}^d)}dt,\end{aligned}$$

where $B^{q,r}_k$ denotes the set of elements of $D(\mathbb{R}^{d+1},\mathbb{C})$ such that the norm $\|\cdot\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))}$ is less than or equal to 1, and U^* is the adjoint operator of U.

Thus, using Cauchy-Schwarz inequality, we deduce that

$$\|U(t)u_0\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))}\leq\|u_0\|_{L^2_k(\mathbb{R}^d)}\sup_{\varphi\in B^{q,r}_k}\left\|\int_{\mathbb{R}}U^*(t)\varphi(t,\cdot)dt\right\|_{L^2_k(\mathbb{R}^d)}.$$

This duality argument simply says that inequality (4.54) implies (4.53). In order to prove (4.54), let us write

$$\begin{aligned}\left\|\int_{\mathbb{R}}U^*(t)\varphi(t,\cdot)dt\right\|^2_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^2}\langle U^*(t)\varphi(t,\cdot),U^*(t')\varphi(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)}dtdt'\\ &= \int_{\mathbb{R}^2}\langle U(t')U^*(t)\varphi(t,\cdot),\varphi(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)}dtdt'.\end{aligned}$$

As $(U(t))_{t\in\mathbb{R}}$ is a bounded family of operators on $L^2_k(\mathbb{R}^d)$, by using the dispersive estimate (4.52) we get, thanks to the interpolation theorem,

$$\|U(t)U^*(t')\varphi(t,\cdot)\|_{L^r_k(\mathbb{R}^d)}\leq\frac{C}{|t-t'|^{\gamma(r)+1}}\|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)},\tag{4.57}$$

for all $r \in [2, \infty]$, where $\gamma(r) = \left(\frac{d}{2} + \gamma\right)\left(1 - \frac{2}{r}\right) - 1$.

In the sharp $\gamma + \frac{d}{2}$-admissible case we have

$$\frac{1}{q'} - \frac{1}{q} = -\gamma(r).$$

The relation (4.57) and Hölder's inequality give

$$\left\| \int_{\mathbb{R}} U^*(t)\varphi(t,\cdot)dt \right\|^2_{L^2_k(\mathbb{R}^d)} \leq \int_{\mathbb{R}^2} \frac{C}{|t-t'|^{\gamma(r)+1}} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dtdt'.$$

We put

$$k(t) = \int_{\mathbb{R}} \frac{1}{|t-t'|^{\gamma(r)+1}} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dt'.$$

Hence

$$\int_{\mathbb{R}^2} \frac{1}{|t-t'|^{\gamma(r)+1}} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dtdt' = \int_{\mathbb{R}} k(t) \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dt.$$

Hölder inequality implies that

$$\int_{\mathbb{R}^2} \frac{1}{|t-t'|^{\gamma(r)+1}} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dtdt'$$
$$\leq \|\varphi\|_{L^{q'}(\mathbb{R}; L^{r'}_k(\mathbb{R}^d))} \left(\int_{\mathbb{R}} |k(t)|^q dt \right)^{\frac{1}{q}}.$$

On the other hand from Hardy-Littlewood-Sobolev inequality we have

$$\int_{\mathbb{R}} |k(t)|^q dt \leq C \|\varphi\|^q_{L^{q'}(\mathbb{R}; L^{r'}_k(\mathbb{R}^d))}.$$

Finally we deduce (4.53).

Assuming the condition (4.56), then (4.57) can be improved to

$$\|U(t)U^*(t')\varphi(t,\cdot)\|_{L^r_k(\mathbb{R}^d)} \leq \frac{C}{(1+|t-t'|)^{\gamma(r)+1}} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)},$$

and now Young's inequality gives the result when

$$\frac{1}{q'} < \frac{1}{q} - \gamma(r),$$

or in other words when (q, r) is nonsharp admissible. This concludes the proof of (4.53), (4.54) when $(q, r) \neq P$. In the same way we prove (4.55).

2nd step: $(q,r) = P$. The idea is use that the estimate

$$\left\| \int_{\mathbb{R}} U^*(t) f(t,\cdot) dt \right\|_{L^2_k(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}; L^{r'}_k(\mathbb{R}^d))} \text{ with } r = \frac{2d+4\gamma}{d+2\gamma-2}$$

is implied by the following fact. If B denotes the bilinear form defined by

$$B(f,g) = \int_{\mathbb{R}^2} \langle U(t')U^*(t) f(t,\cdot), g(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)} dt dt'$$

then B is a continuous on $L^2(\mathbb{R}; L^{r'}_k(\mathbb{R}^d))$.

Let us decompose B into a sum of a simple operators B_j, namely

$$B(f,g) = \sum_{j\in\mathbb{Z}} B_j(f,g)$$

with

$$B_j(f,g) = \int_{2^j \leq |t-t'| \leq 2^{j+1}} \langle U(t')U^*(t) f(t,\cdot), g(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)} dt dt'.$$

The key point of the proof is the following lemma:

Lemma 1. *A neighborhood V of (r^{-1}, r^{-1}) exists such that, for any $(a,b) \in V$, and $j \in \mathbb{Z}$ one holds*

$$|B_j(f,g)| \leq C 2^{-j\beta(a,b)} \|f\|_{L^2(\mathbb{R}; L^{a'}_k(\mathbb{R}^d))} \|g\|_{L^2(\mathbb{R}; L^{b'}_k(\mathbb{R}^d))}$$

with $\beta(a,b) = \left(\frac{d}{2} + \gamma\right)\left(1 - \frac{1}{a} - \frac{1}{b}\right) - 1.$

Proof. Using rescaling, it suffices to prove the lemma for $j = 0$. First of all, using (4.57), we get, for any $a \geq 2$,

$$\begin{aligned} |B_0(f,g)| &\leq C \int_{1\leq|t-t'|\leq 2} \|U(t)U^*(t') f(t,\cdot)\|_{L^a_k(\mathbb{R}^d)} \|g(t',\cdot)\|_{L^{a'}_k(\mathbb{R}^d)} dt dt' \\ &\leq C \int_{1\leq|t-t'|\leq 2} \|f(t,\cdot)\|_{L^{a'}_k(\mathbb{R}^d)} \|g(t',\cdot)\|_{L^{a'}_k(\mathbb{R}^d)} dt dt'. \end{aligned}$$

By Cauchy-Schwarz inequality, we get, for any $a \geq 2$,

$$B_0(f,g) \leq C \|f\|_{L^2(\mathbb{R}; L^{a'}_k(\mathbb{R}^d))} \|g\|_{L^2(\mathbb{R}; L^{a'}_k(\mathbb{R}^d))}. \tag{4.58}$$

Now, let us prove that, for any $a > r$, we have

$$|B_0(f,g)| \leq C \|f\|_{L^2(\mathbb{R}; L^{a'}_k(\mathbb{R}^d))} \|g\|_{L^2(\mathbb{R}; L^2_k(\mathbb{R}^d))}. \tag{4.59}$$

By the definition of B_0, we have

$$\begin{aligned}|B_0(f,g)| &= \int_{1\le|t-t'|\le 2}\langle U^*(t)f(t,\cdot),U^*(t')g(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)}dtdt'\\ &= \int_{\mathbb{R}}\left\langle\int_{\mathbb{R}}U^*(t)f_{t'}(t,\cdot)dt,U^*(t')g(t',\cdot)\right\rangle_{L^2_k(\mathbb{R}^d)}dt'\end{aligned}$$

with

$$f_{t'}(t,\cdot)=1_{\{1\le|t-t'|\le 2\}}(t)f(t,\cdot).$$

Then, applying Cauchy-Schwarz inequality, we obtain

$$|B_0(f,g)|\le\int_{\mathbb{R}}\left\|\int_{\mathbb{R}}U^*(t)f_{t'}(t,\cdot)dt\right\|_{L^2_k(\mathbb{R}^d)}\|g(t',\cdot)\|_{L^2_k(\mathbb{R}^d)}dt'.$$

Using the estimate (4.54) with $(q(a),a)$ sharp $\dfrac{d+2\gamma}{2}$-admissible and $a<r$, we have

$$|B_0(f,g)|\le\int_{\mathbb{R}}\|f_{t'}\|_{L^{q(a)'}(\mathbb{R};L^{a'}_k(\mathbb{R}^d))}\|g(t',\cdot)\|_{L^2_k(\mathbb{R}^d)}dt'.$$

By the definition of $f_{t'}$, this gives, with $F_{a'}(t)=\|f(t,\cdot)\|_{L^{a'}_k(\mathbb{R}^d)}$, that

$$\begin{aligned}|B_0(f,g)| &\le C\int_{\mathbb{R}}\left(\int_{1\le|t-t'|\le 2}F_{a'}(t)^{q(a)'}dt\right)^{\frac{1}{q(a)'}}\|g(t',\cdot)\|_{L^2_k(\mathbb{R}^d)}dt'\\ &\le C\int_{\mathbb{R}}\left(1_{\{1\le|\tau|\le 2\}}*F_{a'}^{q(a)'}\right)^{\frac{1}{q(a)'}}(t')\|g(t',\cdot)\|_{L^2_k(\mathbb{R}^d)}dt'.\end{aligned}$$

Then, Cauchy-Schwarz inequality implies that

$$|B_0(f,g)|\le C\left\|1_{\{1\le|\tau|\le 2\}}*F_{a'}^{q(a)'}\right\|^{\frac{1}{q(a)'}}_{L^{\frac{2}{q(a)'}}(\mathbb{R})}\|g\|_{L^2(\mathbb{R};L^2_k(\mathbb{R}^d))}.$$

As $q(a)'<2$ and $1_{\{1\le|\tau|\le 2\}}\in L^1(\mathbb{R})$, Young's inequality implies that

$$\left\|1_{\{1\le|\tau|\le 2\}}*F_{a'}^{q(a)'}\right\|_{L^{\frac{2}{q(a)'}}(\mathbb{R})}\le C\|F_{a'}\|^{q(a)'}_{L^2(\mathbb{R})}.$$

Thus, inequality (4.59) is proved. The lemma will then follow by interpolation between (4.58) and (4.59). □

End of the proof of Theorem 3. Again we use the atomic decomposition of $f(t,\cdot)$ and $g(t,\cdot)$. Then we have

$$f(t,x)=\sum_{n\in\mathbb{Z}}c_n(t)f_n(t,x) \quad \text{and} \quad g(t,x)=\sum_{m\in\mathbb{Z}}d_m(t)g_m(t,x).$$

Proceeding as in [8] and using that

$$\frac{d}{2}+\gamma-1=\frac{d+2\gamma}{r},$$

we infer for any $(a,b)\in V$,

$$\begin{aligned}|B_j(c_nf_n,d_mg_m)| &\le C\|c_n\|_{L^2(\mathbb{R})}\|d_m\|_{L^2(\mathbb{R})}2^{-j\beta(a,b)}2^{-n(\frac{1}{r'}-\frac{1}{a'})}2^{-m(\frac{1}{r'}-\frac{1}{b'})}\\ &\le 2^{(-j(\frac{d}{2}+\gamma)+n)(\frac{1}{r}-\frac{1}{a})}2^{(-j(\frac{d}{2}+\gamma)+m)(\frac{1}{r}-\frac{1}{b})}\|c_n\|_{L^2(\mathbb{R})}\|d_m\|_{L^2(\mathbb{R})}.\end{aligned}$$

Then, choosing a and b such that

$$\left(-j\left(\frac{d}{2}+\gamma\right)+n\right)\left(\frac{1}{r}-\frac{1}{a}\right)<0 \quad \text{and} \quad \left(-j\left(\frac{d}{2}+\gamma\right)+m\right)\left(\frac{1}{r}-\frac{1}{b}\right)<0,$$

we get that, if $r<+\infty$, then

$$\begin{aligned}|B_j(c_nf_n,d_mg_m)| &\le C\|c_n\|_{L^2(\mathbb{R})}\|d_m\|_{L^2(\mathbb{R})}2^{-2\varepsilon|j(\frac{d}{2}+\gamma)-n|}2^{-2\varepsilon|j(\frac{d}{2}+\gamma)-m|}\\ &\le C\|c_n\|_{L^2(\mathbb{R})}\|d_m\|_{L^2(\mathbb{R})}2^{-\varepsilon|j(\frac{d}{2}+\gamma)-n|}2^{-\varepsilon|n-m|}.\end{aligned}$$

This gives

$$\begin{aligned}|B(f,g)| &\le C\sum_{j,n,m}\|c_n\|_{L^2(\mathbb{R})}\|d_m\|_{L^2(\mathbb{R})}2^{-\varepsilon|j(\frac{d}{2}+\gamma)-n|}2^{-\varepsilon|n-m|}\\ &\le C\sum_{n,m}\|c_n\|_{L^2(\mathbb{R})}\|d_m\|_{L^2(\mathbb{R})}2^{-\varepsilon|n-m|}.\end{aligned}$$

Using weighted Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}|B(f,g)| &\le C\left(\sum_n\|c_n\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}\left(\sum_m\|d_m\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}\\ &\le C\left(\int_R\|(c_n(t))_n\|^2_{l^2}dt\right)^{\frac{1}{2}}\left(\int_R\|(d_m(t))_m\|^2_{l^2}dt\right)^{\frac{1}{2}}.\end{aligned}$$

As $r'<2$, we have $\|(c_n(t))_n\|_{l^2}\le\|(c_n(t))_n\|_{l^{r'}}$. Using the properties of atomic decomposition, we get

$$\begin{aligned}|B(f,g)| &\leq C\left(\int_{\mathbb{R}}\|(c_n(t))_n\|_{l^{r'}}^2 dt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\|(d_m(t))_m\|_{l^{r'}}^2 dt\right)^{\frac{1}{2}}\\ &\leq C\|f\|_{L^2(\mathbb{R};L_k^{r'}\mathbb{R}^d))}\|g\|_{L^2(\mathbb{R};L_k^{r'}(\mathbb{R}^d))}.\end{aligned}$$

Theorem 3 is proved. □

5. Applications.

5.1. Strichartz-type Schrödinger estimates. The main application of Theorem 3 is the following result:

Theorem 4. *Suppose that $d \geq 1$ and that (q,r) and (q_1,r_1) are $\frac{d+2\gamma}{2}$-admissible pairs. If u is a solution to the problem*

$$\left\{\begin{array}{lll}\partial_t u - i\triangle_k u &=& F(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^d\\ u_{|t=0} &=& g\end{array}\right. \tag{5.60}$$

for some data, g, F and time $0 < T < \infty$, then

$$\|u\|_{L^q([0,T];L_k^r(\mathbb{R}^d))} + \|u\|_{C([0,T];L_k^2(\mathbb{R}^d))} \leq C\left(\|g\|_{L_k^2(\mathbb{R}^d)} + \|F\|_{L^{q_1'}([0,T];L_k^{r_1'}(\mathbb{R}^d))}\right). \tag{5.61}$$

Conversely, if the above estimate holds for all g,F,T, then (q,r) and (q_1,r_1) must be $\frac{d+2\gamma}{2}$-admissible.

Remark. i) The case $T=\infty$ in (5.61) can be handled by the usual limiting argument.

ii) We note that the Strichartz estimates for the Dunkl-wave equation have been studied in [11].

Proof. We will prove the sufficient condition first. Indeed we assume that (q,r) satisfy the condition of the theorem, and that u is a solution of (5.60).

We write u as

$$u(t,x) = \mathcal{I}_k(t)g(x) + \int_0^t \mathcal{I}_k(t-s)F(s,x)ds, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \tag{5.62}$$

where $\mathcal{I}_k(t)$ is the unitary operator defined by

$$\mathcal{I}_k(t)(v) := \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}}e^{-i(d+2\gamma)\frac{\pi}{4}sgn\,t}e^{i\frac{\|\cdot\|^2}{4t}}\left[\mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}}v)\right]\left(\frac{\cdot}{2t}\right). \tag{5.63}$$

It is properly defined on $L^1_k(\mathbb{R}^d)$ and $L^2_k(\mathbb{R}^d)$. Below, we note by Φ_k the operator defined by

$$\Phi_k(F)(t,x) := \int_0^t \mathcal{I}_k(t-s)F(s,x)ds. \tag{5.64}$$

The energy estimate

$$\|\mathcal{I}_k(t)g\|_{L^2_k(\mathbb{R}^d)} = \|g\|_{L^2_k(\mathbb{R}^d)}$$

follows from Proposition 8, and the estimate

$$\|\mathcal{I}_k(t-s)g\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{C}{|t-s|^{\gamma+\frac{d}{2}}}\|g\|_{L^1_k(\mathbb{R}^d)}$$

follows from Corollary 1. Replacing the $C([0,T];L^2_k(\mathbb{R}^d))$ norm in the above by the $L^\infty([0,T];L^2_k(\mathbb{R}^d))$ norm, the all estimates will follow from Theorem 3.

We now address the question of continuity in L^2_k. The continuity of $\mathcal{I}_k(\cdot)g$ follows from Proposition 2. To show that the quantity $\Phi_k(F)$ is continuous in L^2_k, one can use the identity

$$\Phi_k(F)(t+\varepsilon) = \mathcal{I}_k(\varepsilon)\Big[\Phi_k(F)(t) + \Phi_k(1_{[t,t+\varepsilon]}F)(t)\Big],$$

the continuity of $\mathcal{I}_k(\varepsilon)$ as an operator on L^2_k, and the fact that

$$\|1_{[t,t+\varepsilon]}F\|_{L^{q_1'}([0,T];L_k^{r_1'}(\mathbb{R}^d))} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We finish the proof of necessity as in [8]. $\square$

Corollary 3. *Let I be an interval of $\mathbb{R}$ (bounded or not). If (q,r) and (q_1,r_1) are $\frac{d+2\gamma}{2}$-admissible pairs, then there exits a constant C such that*

$$\|\mathcal{I}_k(\cdot)g\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))} \leq C\|g\|_{L^2_k(\mathbb{R}^d)},\ \|\Phi_k(F)\|_{L^q(I;L^r_k(\mathbb{R}^d))} \leq C\|F\|_{L^{q_1'}(I;L_k^{r_1'}(\mathbb{R}^d))}.$$

Proposition 11. *If $p \in [2,\infty]$ and $t \neq 0$, then $\mathfrak{I}_k(t)$ maps $L^{p'}_k(\mathbb{R}^d)$ continuously to $L^p_k(\mathbb{R}^d)$ and*

$$\|\mathfrak{I}_k(t)g\|_{L^p_k(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|g\|_{L^{p'}_k(\mathbb{R}^d)}. \tag{5.65}$$

Proof. It follows from Corollary 1 and Proposition 8 that

$$\|\mathfrak{I}_k(t)g\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{1}{c_k|t|^{(\gamma+\frac{d}{2})}}\|g\|_{L^1_k(\mathbb{R}^d)} \quad \text{and} \quad \|\mathfrak{I}_k(t)g\|_{L^2_k(\mathbb{R}^d)} = \|g\|_{L^2_k(\mathbb{R}^d)}.$$

The general case is obtained by interpolation between the cases $p = 2$ and $p = \infty$. □

Proposition 12. *Let I be an interval of $\mathbb{R}$ (bounded or not). Assume* $2 < r < \dfrac{2d+4\gamma}{d+2\gamma-2}$ *($2 < r \leq \infty$ if $d = 1$) and let $(q_1, r_1) \in (1,\infty)^2$ satisfy*

$$\frac{1}{q_1} + \frac{1}{r_1} = (d+2\gamma)\left(\frac{1}{2} - \frac{1}{r}\right).$$

Then $\Phi_k(F) \in L^{q_1}(I; L^r_k(\mathbb{R}^d))$ for every $F \in L^{r_1'}(I; L^{r'}_k(\mathbb{R}^d))$. Moreover, there exits a constant C independent on I such that

$$\|\Phi_k(F)\|_{L^{q_1}(I;L^r_k(\mathbb{R}^d))} \leq C\|F\|_{L^{r_1'}(I;L^{r'}_k(\mathbb{R}^d))}, \quad \textit{for every} \quad F \in L^{r_1'}(I; L^{r'}_k(\mathbb{R}^d)). \tag{5.66}$$

Proof. By density, we need to prove (5.66) for $F \in C_c(I; \mathcal{S}(\mathbb{R}^d))$. It follows from (5.65) that

$$\|\Phi_k(F)(t,\cdot)\|_{L^r_k(\mathbb{R}^d)} \leq \int_0^t \frac{1}{(c_k^2|t-s|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{r})}}\|F(s,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)}ds,$$

and so (5.66) is an immediate consequence of Proposition 10. □

5.2. Well-posedness results for NLDS. In this subsection, Strichartz estimates are a powerful tool to prove local and global well-posedness results for the nonlinear Dunkl-Schrödinger equations. We begin with the model case of the pure power nonlinearity, i.e., we consider a Cauchy problem of the form

$$\begin{cases} \partial_t u(t,x) - i\triangle_k u(t,x) &= F(u(t,x)), \quad (t,x) \in I \times \mathbb{R}^d \\ u_{|t=0} &= g \in L^2_k(\mathbb{R}^d), \end{cases} \tag{5.67}$$

where $d \geq 2$, u is complex-valued function defined on $I \times \mathbb{R}^d$ and the nonlinearity $F \in C(\mathbb{C},\mathbb{C})$ satisfies

$$F(0) = 0, \quad |F(u) - F(v)| \leq C(|u|^p + |v|^p)|u-v|, \quad p > 0. \tag{5.68}$$

We recall the definition of well-posedness.

Definition 7. *We say that the problem* (5.67) *is locally well-posed in* $L^2_k(\mathbb{R}^d)$ *if, for every* g *in* $L^2_k(\mathbb{R}^d)$*, one can find time* $T>0$ *and a unique solution* $u \in C([-T,T], L^2_k(\mathbb{R}^d)) \bigcap X$ *to* (5.67) *which depends continuously on the data, with* X *some additional Banach space. The equation is globally well-posed if these properties hold with* $T=\infty$.

Theorem 5. *If* $p \in \left(0, \dfrac{4}{d+2\gamma}\right]$*, then for every* $g \in L^2_k(\mathbb{R}^d)$*, there exist* $T_{\max}, T_{\min} \in (0,\infty]$ *and a unique, maximal solution* u *of* (5.67) *belonging to*

$$C((-T_{\min}, T_{\max}); L^2_k(\mathbb{R}^d)) \bigcap L^q_{loc}((-T_{\min}, T_{\max}); L^r_k(\mathbb{R}^d))$$

for every sharp $\dfrac{d+2\gamma}{2}$*-admissible pair* (q,r)*. Moreover, the following properties hold:*

(i) There exists $\delta_0>0$ *such that if* $g \in L^2_k(\mathbb{R}^d)$ *satisfies* $\|g\|_{L^2_k(\mathbb{R}^d)} \le \delta_0$*, then the corresponding maximal* L^2_k*-solution is global, i.e.,* $T_{\max} = T_{\min} = \infty$*. Moreover,* u *belongs to* $L^q(\mathbb{R}; L^r_k(\mathbb{R}^d))$ *for every sharp* $\frac{d+2\gamma}{2}$*-admissible pair* (q,r)*.*

(ii) u *depends continuously on* g *in the following sense: If* $g_n \to g$ *in* $L^2_k(\mathbb{R}^d)$ *and if* u_n *denotes the solution of* (5.67) *with* g *replaced by* g_n*, then* $u_n \to u$ *in* $L^q((-S,T), L^r_k(\mathbb{R}^d))$ *for every sharp* $\dfrac{d+2\gamma}{2}$*-admissible pair* (q,r) *and every* S*,* T *satisfying* $-T_{\min} < -S < 0 < T < T_{\max}$*.*

Proof. We proceed in three steps.

Step 1: (Local existence). For the existence, we use a fixed point argument.

• If $p \in \left(0, \dfrac{4}{d+2\gamma}\right)$, fix $T, M>0$ and set

$$X_M := \Big\{ u \in L^q((-T,T); L^{p+2}_k(\mathbb{R}^d)) \bigcap L^\infty((-T,T); L^2_k(\mathbb{R}^d)) : \\ \|u\|_{L^\infty((-T,T);L^2_k(\mathbb{R}^d))} + \|u\|_{L^q((-T,T);L^{p+2}_k(\mathbb{R}^d))} \le M \Big\},$$

where $(q, p+2)$ is sharp $\dfrac{d+2\gamma}{2}$-admissible pair. Note that by Theorem 4 and Corollary 3, this space is never empty. Moreover, it is easily checked that X_M is a complete metric space when equipped with the distance

$$d(u,v) = \|u-v\|_{L^\infty((-T,T);L^2_k(\mathbb{R}^d))} + \|u-v\|_{L^q((-T,T);L^{p+2}_k(\mathbb{R}^d))}.$$

For simplify, we put

$$\|v\|_{X_M} = \|v\|_{L^\infty((-T,T);L^2_k(\mathbb{R}^d))} + \|v\|_{L^q((-T,T);L^{p+2}_k(\mathbb{R}^d))}$$

if $v \in X_M$. Take $g \in L^2_k(\mathbb{R}^d)$. We wish to find conditions on T and M which imply that $\mathcal{H}_k$, given by

$$\mathcal{H}_k(u)(t,\cdot) := \mathcal{I}_k(t)g(\cdot) + \int_0^t \mathcal{I}_k(t-s)F(u(s,\cdot))ds,$$

is a strict contraction on X_M. By our nonlinearity assumption (5.68) and Theorem 4 the following estimate holds

$$\|\mathcal{H}_k(u)\|_{X_M} \leq C\left(\|\mathcal{I}_k(\cdot)g\|_{X_M} + \|u\|^{p+1}_{L^{(p+1)q_1'}((-T,T);L_k^{(p+1)r_1'}(\mathbb{R}^d))}\right). \tag{5.69}$$

with (q_1, r_1) a sharp $\dfrac{d+2\gamma}{2}$-admissible couple.

• If $p \in \left(0, \dfrac{4}{d+2\gamma}\right)$, we take $r_1 = p+2$ and $(q_1 = q, p+2)$ a sharp $\dfrac{d+2\gamma}{2}$-admissible pair such that $q > p+2$. Then applying Corollary 3 and Hölder's inequality in time we obtain

$$\|\mathcal{H}_k(u)\|_{X_M} \leq C\|g\|_{L^2_k(\mathbb{R}^d)} + CT^{\frac{q-p-2}{q}}\|u\|^{p+1}_{L^q((-T,T);L^{p+2}_k(\mathbb{R}^d))}. \tag{5.70}$$

Hence for every $u \in X_M$ one has

$$\|\mathcal{H}_k(u)\|_{X_M} \leq C\|g\|_{L^2_k(\mathbb{R}^d)} + CT^{\frac{q-p-2}{q}}M^{p+1}.$$

Choosing $M = 2C\|g\|_{L^2_k(\mathbb{R}^d)}$, we see that if T is sufficiently small (depending on $\|g\|_{L^2_k(\mathbb{R}^d)}$) then $\mathcal{H}_k(u) \in X_M$ for all $u \in X_M$. Moreover, arguing as above we obtain

$$d(\mathcal{H}_k(u), \mathcal{H}_k(v)) \leq CT^{\frac{q-p-2}{q}}M^p d(u,v), \tag{5.71}$$

for all $u, v \in X_M$. Thus $\mathcal{H}_k$ is a contraction in X_M provided T is small enough, more precisely if $T \leq \left(\dfrac{1}{2CM^p}\right)^{\frac{q}{q-p-2}}$. Hence $\mathcal{H}_k$ has a fixed point u, which is the unique solution of (5.67) in X_M, and there exist $T_{\max}, T_{\min} \in (0,\infty]$ such that u belongs to

$$C((-T_{\min}, T_{\max}); L^2_k(\mathbb{R}^d)) \bigcap L^q_{loc}((-T_{\min}, T_{\max}); L^r_k(\mathbb{R}^d))$$

for the sharp $\dfrac{d+2\gamma}{2}$-admissible pair $(q, p+2)$, with

$$T_{\max} = \sup\{T > 0, \text{there exists a solution of (5.67) on } [0, T]\},$$

$$T_{\min} = \sup\{T > 0, \text{there exists a solution of (5.67) on } [-T, 0]\}.$$

Moreover, from Theorem 4 and by the argument we use to prove (5.70), it is easy to see that $u \in L^{q_1}_{loc}((-T_{\min}, T_{\max}); L^{r_1}_k(\mathbb{R}^d))$ for every sharp $\dfrac{d+2\gamma}{2}$-admissible pair (q_1, r_1).

• If $p = \dfrac{4}{d+2\gamma}$, let $g \in L^2_k(\mathbb{R}^d)$. Since $\mathcal{I}_k(\cdot)g \in L^{p+2}(\mathbb{R}; L^{p+2}_k(\mathbb{R}^d))$, by Corollary 3, we have

$$\|\mathcal{I}_k(.)g\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))} \to 0 \quad \text{as} \quad T \downarrow 0. \tag{5.72}$$

Therefore there exist $M, T > 0$ such that

$$\|\mathcal{I}_k(.)g\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))} < M. \tag{5.73}$$

Let us consider the set

$$X_M := \Big\{u \in L^{p+2}((-T,T); L^{p+2}_k(\mathbb{R}^d)) : \|u\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))} \leq 2M\Big\}.$$

It is easily checked that X_M is a complete metric space when equipped with the distance

$$d(u,v) = \|u - v\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))}.$$

As above, by Theorem 4, the following estimate holds

$$\begin{aligned}
&\|\mathcal{H}_k(u)\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))}\\
&\qquad \leq C\Big(\|\mathcal{I}_k(.)g\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))} + \|u\|^{p+1}_{L^{(p+2)}((-T,T);L^{(p+2)}_k(\mathbb{R}^d))}\Big),
\end{aligned}$$

where we have taken $q = q_1 = r = r_1 = p+2$. Hence, for every $u \in X_M$:

$$\|\mathcal{H}_k(u)\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))} \leq C\|\mathcal{I}_k(\cdot)g\|_{L^{p+2}((-T,T);L^{p+2}_k(\mathbb{R}^d))} + CM^{p+1}.$$

From relations (5.72) and (5.73) above, we see that if T is small enough, then we can choose M such that $\mathcal{H}_k(u)$ belongs to X_M for all $u \in X_M$. As above we prove also that $\mathcal{H}_k$ is a contraction on the space X_M provided T is sufficiently small. Thus $\mathcal{H}_k$ has a fixed point u, which is the unique solution of (5.67) in X_M.

Moreover, from Theorem 4 it is easy to see that there exist $T_{\max}, T_{\min} \in (0,\infty]$ such that

$$u \in C((-T_{\min}, T_{\max}); L^2_k(\mathbb{R}^d)) \bigcap L^{q_1}_{loc}((-T_{\min}, T_{\max}); L^{r_1}_k(\mathbb{R}^d)),$$

for every sharp $\dfrac{d+2\gamma}{2}$-admissible pair (q_1, r_1).

Step 2: (Uniqueness). We first note that the uniqueness is a local property, so that we need only to establish it on possibly small intervals. To see this, we argue for positive times, the case for negative times being the same. Suppose that $u_1, u_2 \in C([0,T]; L^2_k(\mathbb{R}^d)) \bigcap L^q_{loc}((0,T); L^r_k(\mathbb{R}^d))$ are two solutions of (5.67) and assume that $u_1(t) \neq u_2(t)$ for some $t \in [0,T]$. Let $t_0 = \inf \Big\{t \in [0,T],\ u_1(t) \neq u_2(t)\Big\}$. Since both u_1 and u_2 are continuous into $L^2_k(\mathbb{R}^d)$, this definition makes sense and $u_1(t_0) = u_2(t_0) = \chi$. Moreover, the curves $U_1(t) = u_1(t+t_0)$ and $U_2(t) = u_2(t+t_0)$ both satisfy the equation $w = \mathcal{I}_k(\cdot)\chi + \Phi_k(F(w))$ on $[0, T-t_0]$. As above we apply Theorem 4 and the argument of proof of (5.70), to obtain that for all $t \in [t_0, T]$,

$$\begin{aligned} &\|u_1 - u_2\|_{L^q((t_0,t);L^{p+2}_k(\mathbb{R}^d))} \\ &\quad\leq C(t-t_0)^{\frac{4-(d+2\gamma)p}{4}} \sum_{i=1}^{2} \|u_i\|^p_{L^q((t_0,t);L^{p+2}_k(\mathbb{R}^d))} \|u_1 - u_2\|_{L^q((t_0,t);L^{p+2}_k(\mathbb{R}^d))}, \end{aligned}$$

where $\left(q = \dfrac{4(p+2)}{p(d+2\gamma)}, p+2\right)$ is a sharp $\dfrac{d+2\gamma}{2}$-admissible pair. For $t > t_0$, but sufficiently close to t_0, it follows that

$$C(t-t_0)^{\frac{4-(d+2\gamma)p}{4}} \sum_{i=1}^{2} \|u_i\|^p_{L^q((t_0,t);L^{p+2}_k(\mathbb{R}^d))} < 1,$$

and so that $\|u_1 - u_2\|_{L^q((t_0,t);L^{p+2}_k(\mathbb{R}^d))} = 0$. This contradicts the choice of t_0, and thus proves that $u_1(t) = u_2(t)$ for all $t \in [0,T]$.

Step 3: (Global existence). As above, by Theorem 4 the following Strichartz estimate holds

$$\|\mathcal{H}_k(u)\|_X \leq C\Big(\|g\|_{L^2_k(\mathbb{R}^d)} + \|F(u)\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))}\Big), \tag{5.74}$$

with $(q, r = p+2)$ a sharp $\dfrac{d+2\gamma}{2}$-admissible and
$X = C(\mathbb{R}; L^2_k(\mathbb{R}^d)) \bigcap L^q(\mathbb{R}; L^r_k(\mathbb{R}^d))$ the Banach space with norm

$$\|v\|_X = \|v\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))} + \|v\|_{C(\mathbb{R};L^2_k(\mathbb{R}^d))}.$$

By our nonlinearity assumption (5.68) and Hölder inequality we have

$$\|\mathcal{H}_k(u)\|_X \leq C\left(\|g\|_{L^2_k(\mathbb{R}^d)} + \|u\|^{p+1}_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))}\right). \tag{5.75}$$

As above, we have proved that $\mathcal{H}_k$ maps the Banach space X into itself, and moreover the ball X_M into itself, provided M and $\|g\|_{L^2_k(\mathbb{R}^d)}$ are small enough, where

$$X_M = \Big\{u \in X : \|u\|_X < M\Big\}.$$

We assume now that $u_i \in X$ is such that

$$\|u_i\|_X < M,$$

with M small enough, and also that $\|g\|_{L^2_k(\mathbb{R}^d)} < \delta$. By (5.75) we note that

$$\|\mathcal{H}_k(u)\|_X \leq C\delta + CM^{p+1} < M,$$

provided M, δ are such that $CM^p < \dfrac{1}{2}$ and $C\delta < \dfrac{M}{2}$. We have also

$$\begin{aligned}
\|\mathcal{H}_k(u_1) - \mathcal{H}_k(u_2)\|_X &\leq C\|F(u_1) - F(u_2)\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))} \\
&\leq C\|u_1 - u_2\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))}\left(\|u_1\|^p_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))} + \|u_2\|^p_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))}\right) \\
&\leq \|u_1 - u_2\|_{X_M} 2CM^p \leq \frac{1}{2}\|u_1 - u_2\|_{X_M}
\end{aligned}$$

provided M is so small that $2CM^p < \dfrac{1}{2}$. Thus, if initial data are small enough i.e. $\|g\|_{L^2_k(\mathbb{R}^d)} < \delta$, then the map $\mathcal{H}_k$ is a contraction and this implies that there exists a unique solution $u(t,x)$ of the Cauchy problem (5.67) such that $u(t,x) \in L^q(\mathbb{R}; L^r_k(\mathbb{R}^d))$ with a couple (q,r) which is sharp $\dfrac{d+2\gamma}{2}$-admissible pair when $1 < p \leq \dfrac{4}{d+2\gamma}$. As observed above one can see easily that this is the unique solution in $u(t,x) \in C(R, L^2_k(\mathbb{R}^d))$ with small initial data in L^2_k . Thus we have proved the global existence as claimed. The proof of the continuous dependence uses the same idea as in [9]. □

Proposition 13 (Continuation principle)**.** *We assume that F is as in Theorem* 5*. If $g \in L^2_k(\mathbb{R}^d)$ and if u is the maximal solution of* (5.67)*, then we have:*

i) *If $p \in \left(0, \dfrac{4}{d+2\gamma}\right)$ and $T_{\max} < \infty$ (respectively, $T_{\min} < \infty$), then $\|u(t)\|_{L^2_k(\mathbb{R}^d)} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow T_{\min}$).*

ii) *If* $p = \dfrac{4}{d+2\gamma}$ *and* $T_{\max} < \infty$ (*respectively,* $T_{\min} < \infty$), *then* $\|u\|_{L^q((0,T_{\max});L^r_k(\mathbb{R}^d))} = \infty$ (*respectively,* $\|u\|_{L^q((-T_{\min},0);L^r_k(\mathbb{R}^d))} = \infty$) *for every sharp* $\dfrac{d+2\gamma}{2}$*-admissible pair* (q,r) *with* $r \geq p+2$.

Proof. i) If $p \in \left(0, \dfrac{4}{d+2\gamma}\right)$, it follows from Step 1 of the proof of Theorem 5 and the uniqueness property that $T_{\max} - t \geq \left(\dfrac{1}{4C^2\|u(t,\cdot)\|^p_{L^2_k(\mathbb{R}^d)}}\right)^{\frac{q}{q-p-2}}$.

Suppose now that $T_{\max} < \infty$, then $\|u(t)\|_{L^2_k(\mathbb{R}^d)} \geq \left(\dfrac{1}{4C^2(T_{\max}-t)^{\frac{q-p-2}{q}}}\right)^{\frac{1}{p}}$. As $q > p+2$, we obtain $\|u(t)\|_{L^2_k(\mathbb{R}^d)} \to \infty$ as $t \uparrow T_{\max}$. One shows by the same argument that if $T_{\min} < \infty$, then $\|u(t)\|_{L^2_k(\mathbb{R}^d)} \to \infty$ as $t \downarrow T_{\min}$.

ii) If $p = \dfrac{4}{d+2\gamma}$, we show the blowup alternative by contradiction. Suppose that $T_{\max} < \infty$ and that $\|u\|_{L^{(p+2)}((0,T_{\max});L^{(p+2)}_k(\mathbb{R}^d))} < \infty$. Let $0 \leq t \leq t+\tau < T_{\max}$. It follow that

$$\mathcal{I}_k(\tau)u(t,\cdot) = u(t+\tau,\cdot) - \int_0^\tau \mathcal{I}_k(\tau-s)F(u(t+s,\cdot))ds.$$

By Theorem 4 we deduce that there exists C such that

$$\begin{aligned}\|\mathcal{I}_k(\cdot)u(t)\|_{L^{(p+2)}((0,T_{\max}-t);L^{(p+2)}_k(\mathbb{R}^d))} &\leq \|u\|_{L^{(p+2)}((t,T_{\max});L^{(p+2)}_k(\mathbb{R}^d))} \\ &+ C\|u\|^{p+1}_{L^{(p+2)}((t,T_{\max});L^{(p+2)}_k(\mathbb{R}^d))}.\end{aligned}$$

Therefore for t close enough to $T_{\max}$,

$$\|\mathcal{I}_k(\cdot)u(t)\|_{L^{(p+2)}((0,T_{\max}-t);L^{(p+2)}_k(\mathbb{R}^d))} \leq \frac{M}{2}.$$

By Step 1 in the proof of Theorem 5, u can be extended beyong $T_{\max}$, which is a contradiction. This shows that $\|u\|_{L^{(p+2)}((0,T_{\max});L^{(p+2)}_k(\mathbb{R}^d))} = \infty$. Let now (q,r) be a sharp $\dfrac{d+2\gamma}{2}$-admissible pair such that $r > p+2$. It follows from Hölder's inequality that for any $T < T_{\max}$,

$$\|u\|_{L^{(p+2)}((0,T);L^{(p+2)}_k(\mathbb{R}^d))} \leq \|u\|^\lambda_{L^\infty((0,T);L^2_k(\mathbb{R}^d))}\|u\|^{1-\lambda}_{L^q((0,T);L^r_k(\mathbb{R}^d))},$$

with $\lambda = \dfrac{2(r-p-2)}{(p+2)(r-2)}$. Letting $T \uparrow T_{\max}$, we obtain $\|u\|_{L^q((0,T_{\max});L^r_k(\mathbb{R}^d))} = \infty$.

One shows by the same argument that if $T_{\min} < \infty$, then $\|u\|_{L^q((-T_{\min},0);L^r_k(\mathbb{R}^d))} = \infty$. $\square$

Corollary 4. *Let F be as above and $Re(\overline{u}F(u)) = 0$. Then in the subcritical case $0 < p < \dfrac{4}{d+2\gamma}$ the problem* (5.67) *is globally well-posed for arbitrary L^2_k data.*

Proof. A simple calculation shows the conservation of charge (i.e., $\|u(t)\|_{L^2_k(\mathbb{R}^d)} = \|g\|_{L^2_k(\mathbb{R}^d)}$ for $|t| \leq T$). Replacing g by $u(t_0)$ for any $t_0 \in (-T_{\min}, T_{\max})$, we obtain that $\|u(t)\|_{L^2_k(\mathbb{R}^d)}$ is locally constant, hence constant. The global existence follows from the blowup alternative. $\square$

5.3. Scattering for NLDS. In this subsection, we consider the nonlinear Dunkl-Schrödinger equation

$$\partial_t u(t,x) - i\triangle_k u(t,x) = F(u(t,x)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{5.76}$$

with F satisfying (5.68). Let $t_0 \in \overline{\mathbb{R}}$, and consider (5.76) together with the initial data:

$$\mathcal{I}_k(-t)u(t)_{|t=t_0} = g. \tag{5.77}$$

We use the convention: if $t_0 = +\infty$ (resp. $t_0 = -\infty$), then we denote $g = u_+$ (resp. $g = u$), and solving (5.76)-(5.77) means that we construct wave operators. If $t_0 = 0$, then we denote $g = u_0$, and (5.76)–(5.77) is the standard Cauchy problem.

In all these cases, we seek mild solutions to (5.76)–(5.77), that is, we solve

$$u(t,\cdot) = \mathcal{I}_k(t)g(\cdot) + \int_{t_0}^{t} \mathcal{I}_k(t-s)F(u(s,\cdot))ds =: \mathcal{A}_k(u)(t). \tag{5.78}$$

Actually, using our general methods [5], it is possible to prove scattering for small $L^2_k(\mathbb{R}^d)$ data. Thus we can prove:

Proposition 14. *Let $d \geq 2$, $p = \dfrac{4}{d+2\gamma}$ and $t_0 \in \overline{\mathbb{R}}$. There exists δ_0 such that if $g \in L^2_k(\mathbb{R}^d)$ with $\|g\|_{L^2_k(\mathbb{R}^d)} < \delta_0$, then the unique global solution $u(t)$ has the scattering property: there exists $u_\pm \in L^2_k(\mathbb{R}^d)$ such that*

$$\|u(t) - \mathcal{I}_k(t)u_\pm\|_{L^2_k(\mathbb{R}^d)} \to 0 \quad \text{as } t \to \pm\infty.$$

If $t_0 = +\infty$ (resp. $t_0 = -\infty$), then $u_+ = g$ (resp. $u_- = g$).

P r o o f. According to the proof of Theorem 5, for $p = \dfrac{4}{d+2\gamma}$ and small initial data in L^2_k, there exists a unique solution u of (5.76)–(5.77) such that $u \in C(\mathbb{R}; L^2_k(\mathbb{R}^d)) \bigcap L^q(\mathbb{R}; L^r_k(\mathbb{R}^d))$ with a couple (q,r) which is sharp $\dfrac{d+2\gamma}{2}$-admissible. Scattering then follows from the Cauchy criterion. Indeed, using Theorem 4, for $t_1 \leq t_2$, we have

$$\begin{aligned} \|\mathcal{I}_k(-t_2)u(t_2,.) - \mathcal{I}_k(-t_1)u(t_1,\cdot)\|_{L^2_k(\mathbb{R}^d)} &= \left\| \int_{t_1}^{t_2} \mathcal{I}_k(-s)F(u(s,\cdot))ds \right\|_{L^2_k(\mathbb{R}^d)} \\ &\leq C\|u\|^{p+1}_{L^q([t_1,t_2];L^r_k(\mathbb{R}^d))} \end{aligned}$$

for all (q,r) sharp $\dfrac{d+2\gamma}{2}$-admissible. Notice that the right-hand side goes to zero when $t_1, t_2 \to \pm\infty$ since $u \in L^q(\mathbb{R}; L^r_k(\mathbb{R}^d))$ for all couples (q,r) therefore sharp $\dfrac{d+2\gamma}{2}$-admissible. The result follows easily, since the group $\mathcal{I}_k$ is unitary on $L^2_k(\mathbb{R}^d)$. $\square$

In the following we prove that if $F(u) = -iu|u|^p$ with $0 < p \leq \dfrac{2}{d+2\gamma}$ then wave operator does not exist.

Proposition 15. *Let $d \geq 2$, $0 < p \leq \dfrac{2}{d+2\gamma}$ and $T > 0$. Let $u \in C(]-\infty, -T]; L^2_k(\mathbb{R}^d))$ be a solution of* (5.76) *such that there exists $u_- \in L^2_k(\mathbb{R}^d)$ and*

$$\|u(t) - \mathcal{I}_k(t)u_-\|_{L^2_k(\mathbb{R}^d)} \to 0 \quad as\, t \to -\infty.$$

Then $u \equiv 0$ and $u_- \equiv 0$.

P r o o f. Let $\chi \in D(\mathbb{R}^d)$ and $t_1 \leq t_2 \leq -T$: by assumption

$$\begin{aligned} \langle \chi, \mathcal{I}_k(-t_2)u(t_2,.) - \mathcal{I}_k(-t_1)u(t_1,\cdot)\rangle &= -i\left\langle \chi, \int_{t_1}^{t_2} \mathcal{I}_k(-s)|u|^p u ds \right\rangle \\ &= -i\int_{t_1}^{t_2} \langle \mathcal{I}_k(s)\chi, |u|^p u\rangle ds \end{aligned}$$

goes to zero as $t_1, t_2 \to -\infty$. But for $s \to -\infty$ we have

$$\begin{aligned} \mathcal{I}_k(s)\chi &\sim C(k)\frac{e^{\frac{i\|x\|^2}{4s}}}{|s|^{\gamma+\frac{d}{2}}}\mathcal{F}_D(\chi)\left(\frac{x}{2s}\right), \\ u(s) &\sim \mathcal{I}_k(s)u_- \sim C(k)\frac{e^{\frac{i\|x\|^2}{4s}}}{|s|^{\gamma+\frac{d}{2}}}\mathcal{F}_D(u_-)\left(\frac{x}{2s}\right). \end{aligned}$$

Therefore,

$$\langle \mathcal{I}_k(s)\chi, |u|^p u\rangle \sim \frac{C(k)}{|s|^{(\gamma+\frac{d}{2})p}}\langle \mathcal{F}_D(\chi), |\mathcal{F}_D(u_-)|^p \mathcal{F}_D(u_-)\rangle.$$

This function of s is not integrable, unless

$$\langle \mathcal{F}_D(\chi), |\mathcal{F}_D(u_-)|^p \mathcal{F}_D(u_-)\rangle = 0.$$

Since $\chi \in D(\mathbb{R}^d)$ is arbitrary, this means that $\mathcal{F}_D(u_-) \equiv 0 \equiv u_-$. The assumption and the conservation of mass then imply $u \equiv 0$. $\square$

REFERENCES

[1] T. Cazenave, F. Weissler. Some remarks on the nonlinear Schrödinger equation in the critical case. Lect. Notes in Math. vol. **1394**, Springer-Verlag, Berlin, 1989, 18–29.

[2] C. F. Dunkl. Differential-difference operators associated to reflection group. *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[3] C. F. Dunkl. Integral kernels with reflection group invariant. *Canad. J. Math.* **43** (1991), 1213–1227.

[4] C. F. Dunkl. Hankel transforms associated to finite reflection groups. *Contemp. Math.* **138** (1992), 123–138.

[5] J. Ginibre, G. Velo. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case.*J. Funct. Anal.* **32**, *1* (1979), 1–32; II. Scattering theory, general case. *J. Funct. Anal.* **32**, *1* (1979), 33–71.

[6] J. Ginibre, G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.* **133**, *1* (1995), 50–68.

[7] M. F. E. de Jeu. The Dunkl transform. *Invent. Math.* **113** (1993), 147–162.

[8] M. Keel, T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.* **120** (1998), 955–980.

[9] T. Kato. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* **46** (1987), 113–129.

[10] H. Mejjaoli, K. Trimèche. Hypoellipticity and hypoanaliticity of the Dunkl Laplacian operator. *Integral Transform. Spec. Funct.* **15**, *6* (2004), 523–548.

[11] H. Mejjaoli. Strichartz estimates for the Dunkl wave equation and applications. *J. Math. Anal. Appl.* **346**, *1* (2008), 41–54.

[12] M. Rösler. Positivity of Dunkl's intertwining operator. *Duke. Math. J.* **98** (1999), 445–463.

[13] M. Rösler. A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc.* **355** (2003), 2413–2438.

[14] I. E. Segal. Space-time decay for solutions of wave equations. *Adv. Math.* **22** (1976), 304–311.

[15] R. S. Strichartz. Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.* **44** (1977), 705–774.

[16] S. Thangavelu, Y. Xu. Convolution operator and maximal functions for Dunkl transform. *J. Anal. Math.* **97** (2005), 25–56.

[17] K. Trimèche. Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual. *Integral Transform. Spec. Funct.* **12**, *4* (2001), 349–374.

[18] K. Trimèche. Paley-Wiener theorems for Dunkl transform and Dunkl translation operators. *Integral Transform. Spec. Funct.* **13** (2002), 17–38.

[19] K. Yajima. Existence of solutions for Schrödinger evolution equations. *Comm. Math. Phys.* **110** (1987), 415–426.

H. Mejjaoli
Department of Mathematics
Faculty of Sciences of Tunis
CAMPUS 1060 Tunis, Tunisia
e-mail: hatem.mejjaoli@ipest.rnu.tn

Received August 18, 2008
Revised December 12, 2008